Logarithmic Time Prediction

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(With help from Alina Beygelzimer & Bianca Zadrozny)

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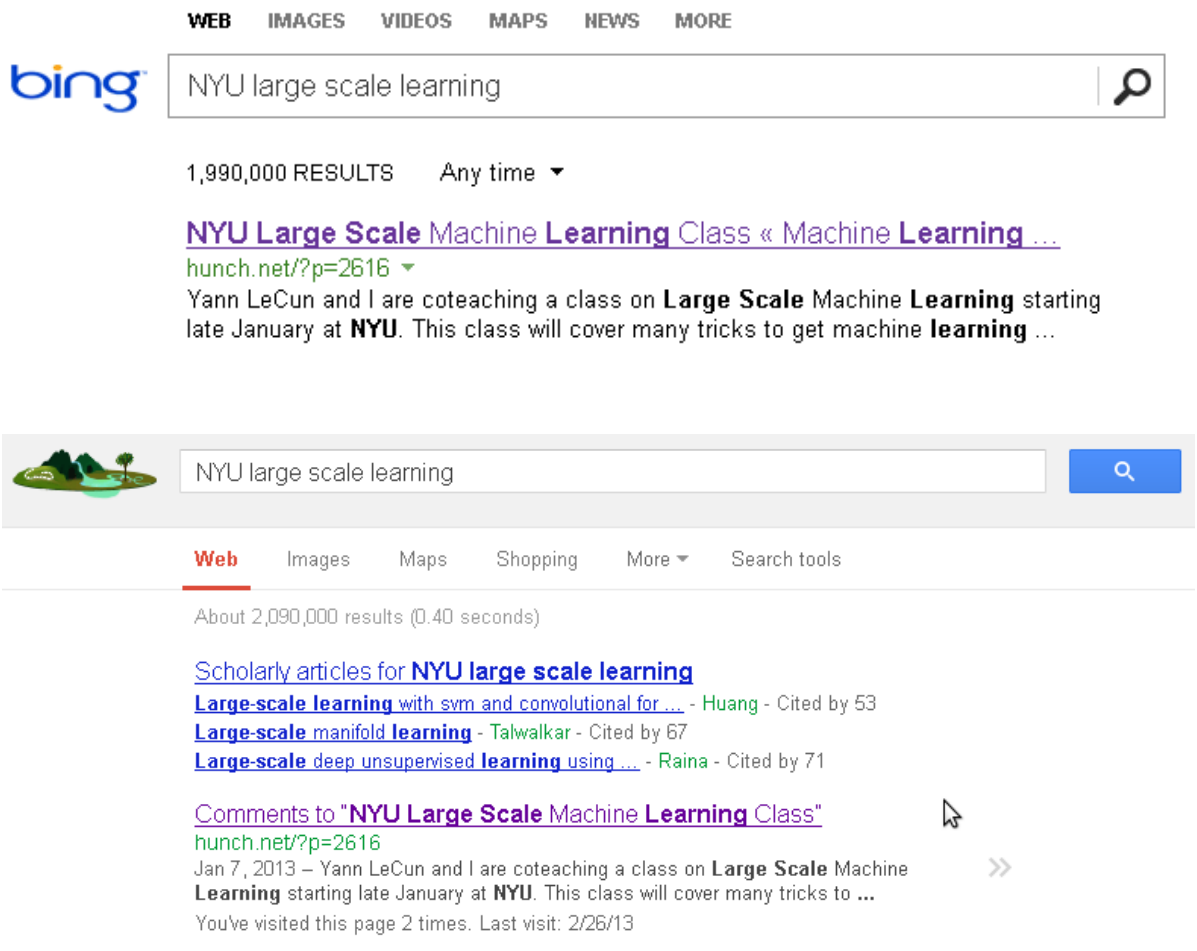
A Scenario

You have 10^{10} webpages and want to return the best result in 100ms.



How do you do it?

Method 1: Indexing algorithms

1. Inverted Index
2. Weighted And (WAND)
3. Locality Sensitive Hashing
4. Predictive Indexing and other Data-driven variations of the above

See previous lecture on indexing.

Method 2: Logarithmic time prediction

Typical time to choose one of k things with ML algorithms: $O(k)$.

Minimum possible time to choose one of k things: $O(\log k)$.

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How can we make algorithms which run in $O(\log k)$ time?

Why logarithmic time prediction?

Hypothesis: Indexing + scoring on top candidates is unsound.
Evidence:

1. If you apply a learned quality predictor to all results (on a small subset of queries) you get different results.
2. Machine Learning (well) is hard. Indexing (well) is hard. Getting the two to work together (well) is hard. Maybe a master algorithm can do both better?
3. Common story line for an ML paper: We figured out how to do end-to-end learning, and it worked better.

How do we do logarithmic time prediction?

Approach 1: Build a decision tree with $O(\log k)$ depth.

Difficulties:

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Solution:

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Solution: Use map-reduce with 1000 nodes to train a decision forest.

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It works! And beats an index+score approach. See WWW 2013 paper.

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Approach 2: Reuse existing algorithms multiple times in a log-depth structure.

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Existing algorithms solve:

Binary Classification

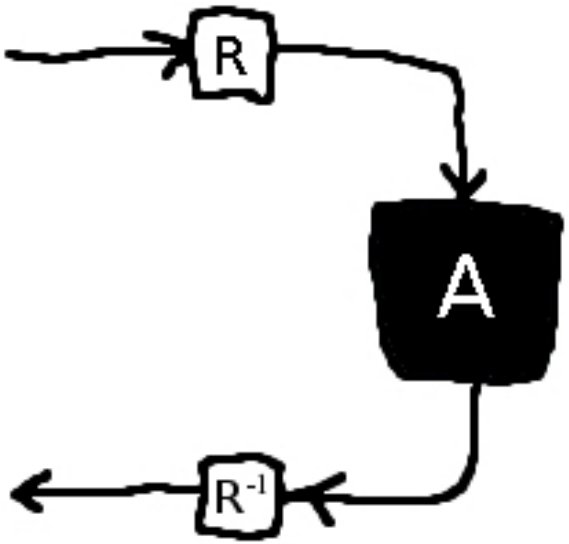
- Given training data $\{(\mathbf{x_1}, y_1), \ldots, (\mathbf{x_n}, y_n)\}$, produce a classifier $h : X \to \{0, 1\}$.
- Unknown underlying distribution D over $X \times \{0, 1\}$.
- Find h with small 0-1 loss:

$$\ell_{0/1}(h, D) = \mathbf{Pr}_{(x,y)\sim D}[h(x) \neq y]$$

Years of research and development of algorithms for solving this problem. Can we reuse it?

Learning Reductions

Goal: minimize ℓ on D



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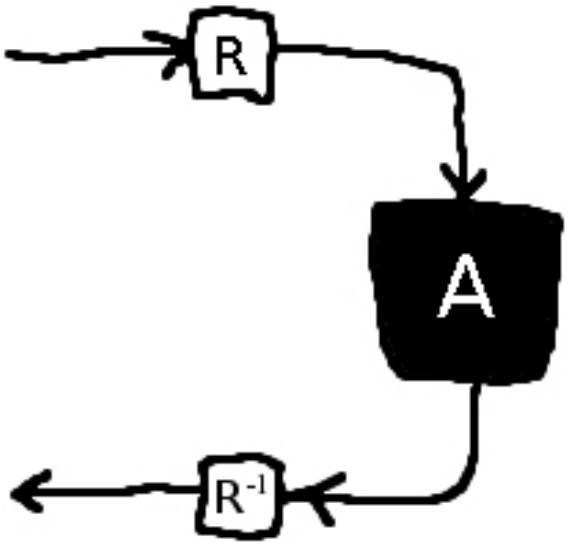

Algorithm for optimizing $\ell_{0/1}$

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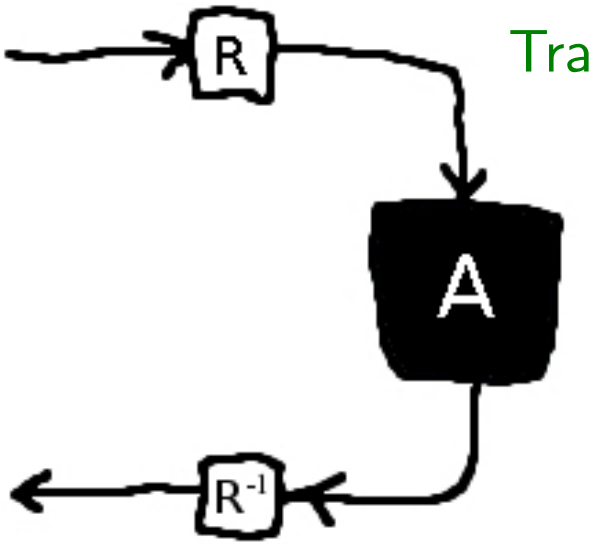

Transform D into D_R

Algorithm for optimizing $\ell_{0/1}$

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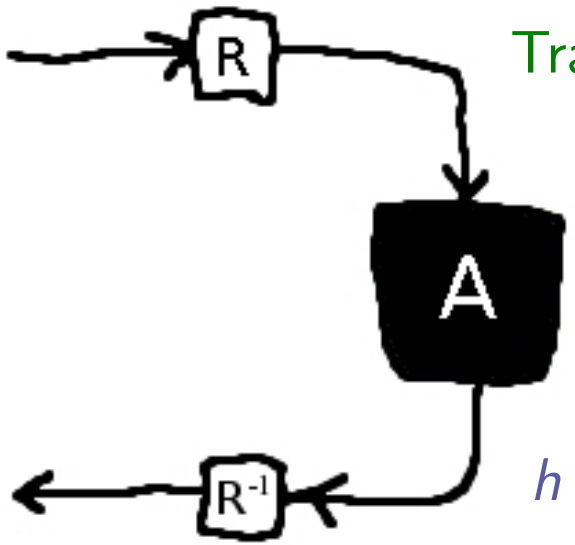


Transform D into D_R

Algorithm for optimizing $\ell_{0/1}$

h

Transform h with small $\ell_{0/1}(h, D_R)$ into R_h with small $\ell(R_h, D)$.

such that if h does well on $(D_R, \ell_{0,1})$, R_h is guaranteed to do well on (D, ℓ).

Let's begin with ...

Multi-class classification

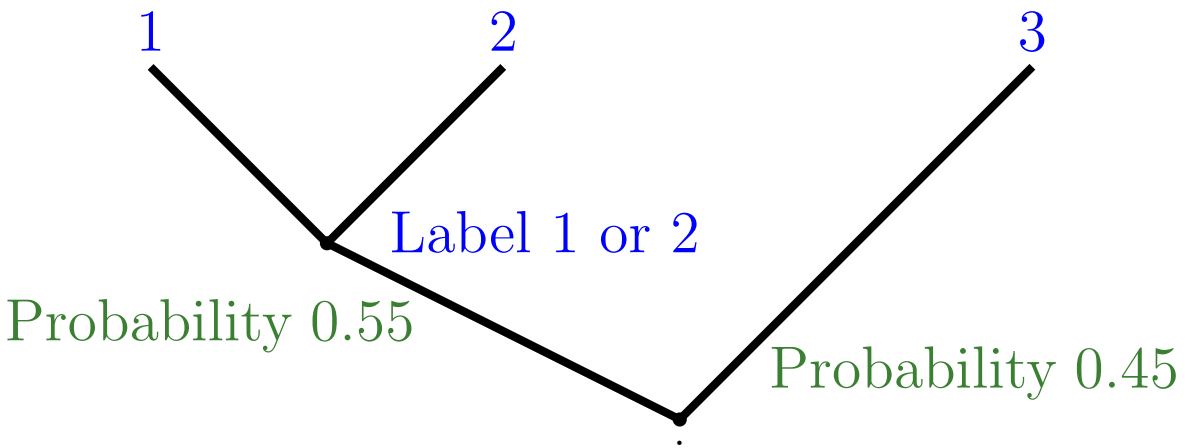
Distribution D over $X \times Y$, where $Y = \{1, \ldots, k\}$.

Find a classifier $h : X \to Y$ minimizing the multi-class loss on D

$$\ell_k(h, D) = \mathbf{Pr}_{(x,y)\sim D}[h(x) \neq y]$$

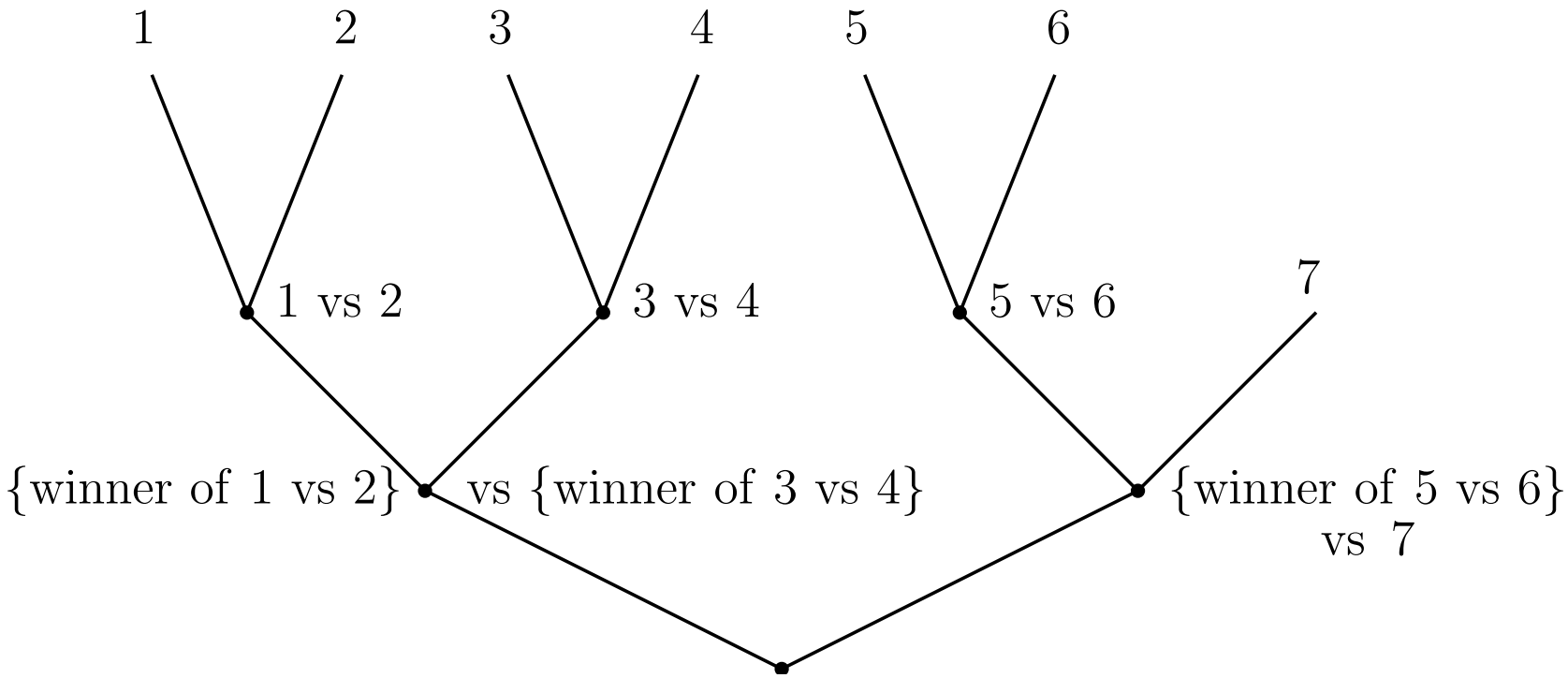
Caution: Trees go wrong

3 choices: $P(1 \mid x) = P(2 \mid x) = 0.275$, $P(3 \mid x) = 0.45$.



Optimal classifier picks 1 or 2, but choice 3 is best.

The Filter Tree (Single-Elimination Tournament)



Each non-leaf predicts the best of a pair of winners from the previous round

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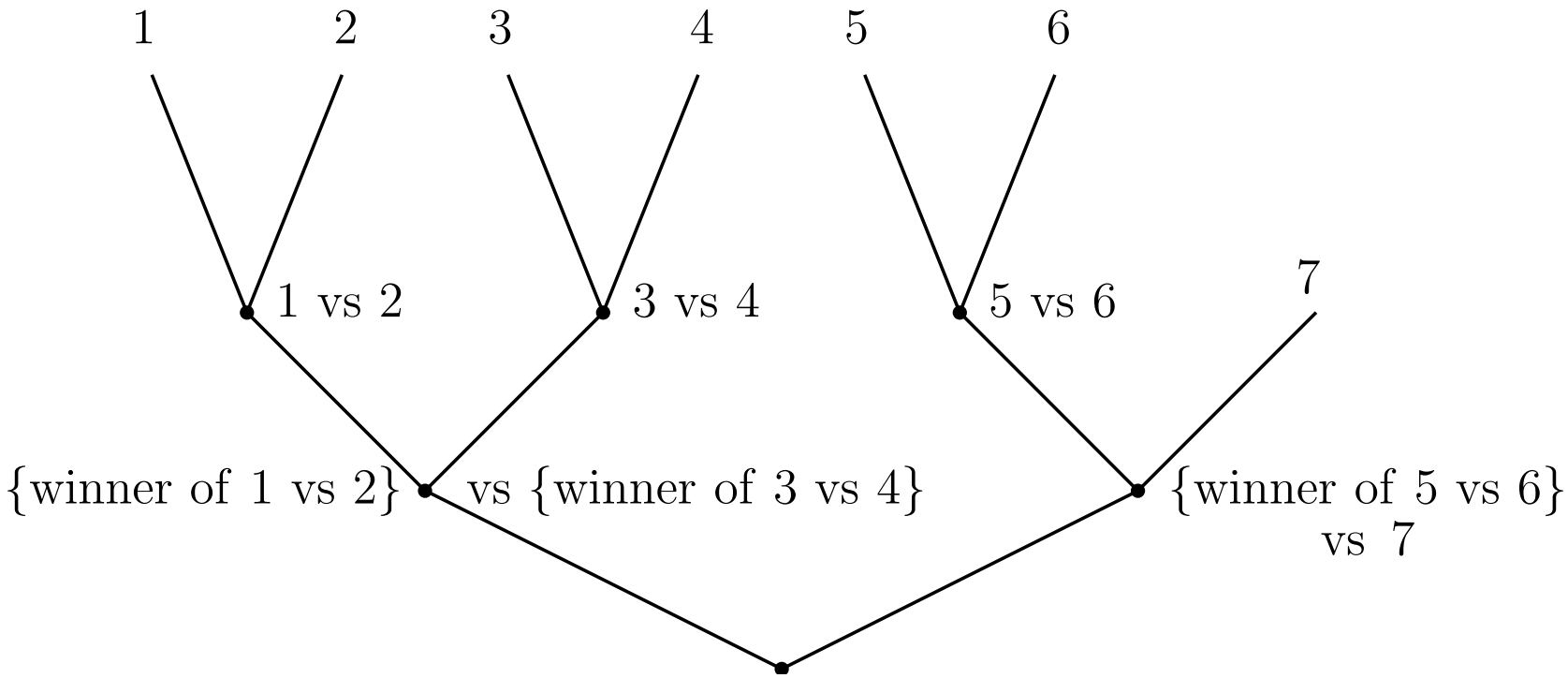

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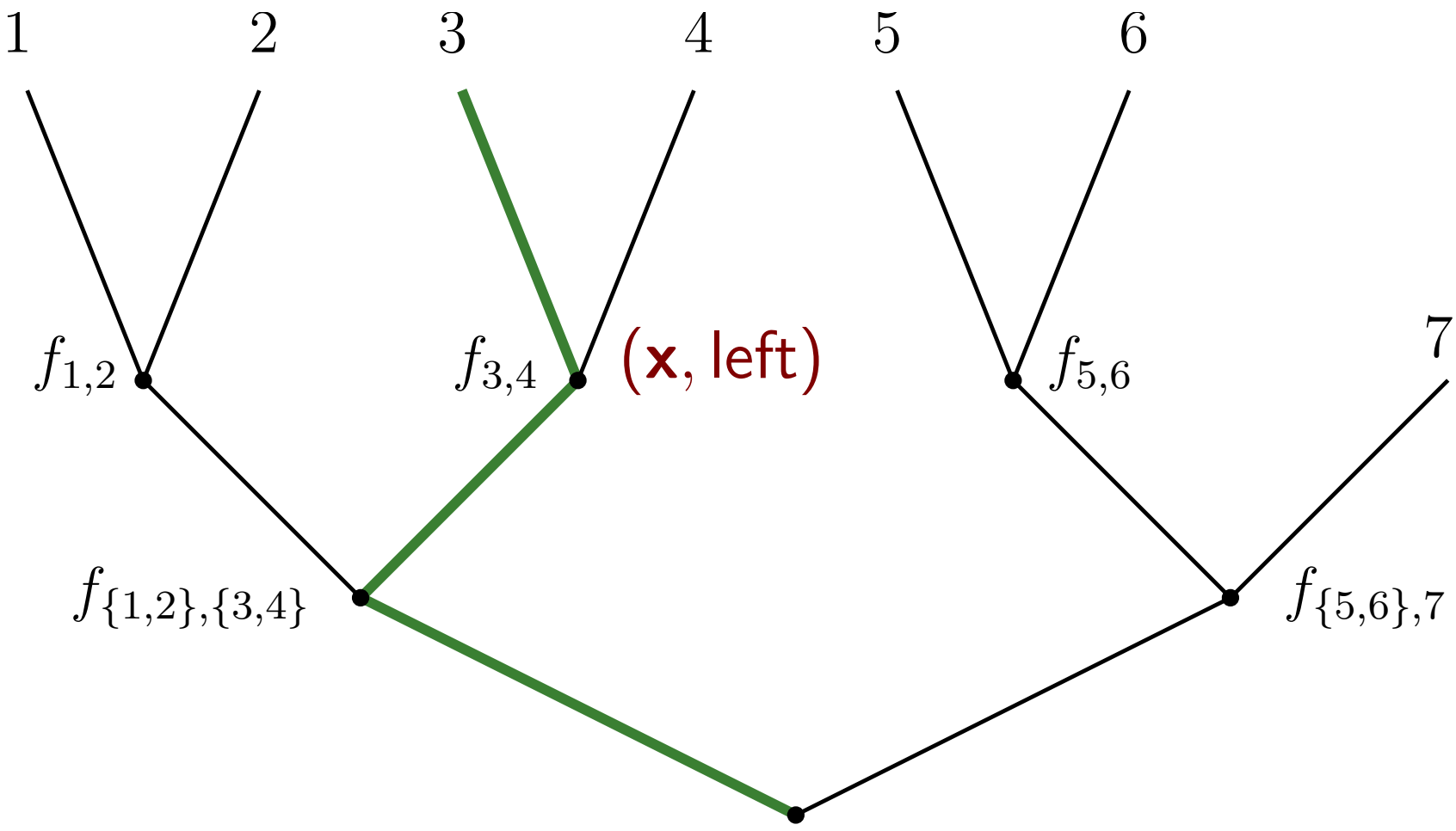
To predict on x: follow the chain of predictions from root to leaf, output the leaf.

Training on example $(\mathbf{x}, \mathbf{3})$



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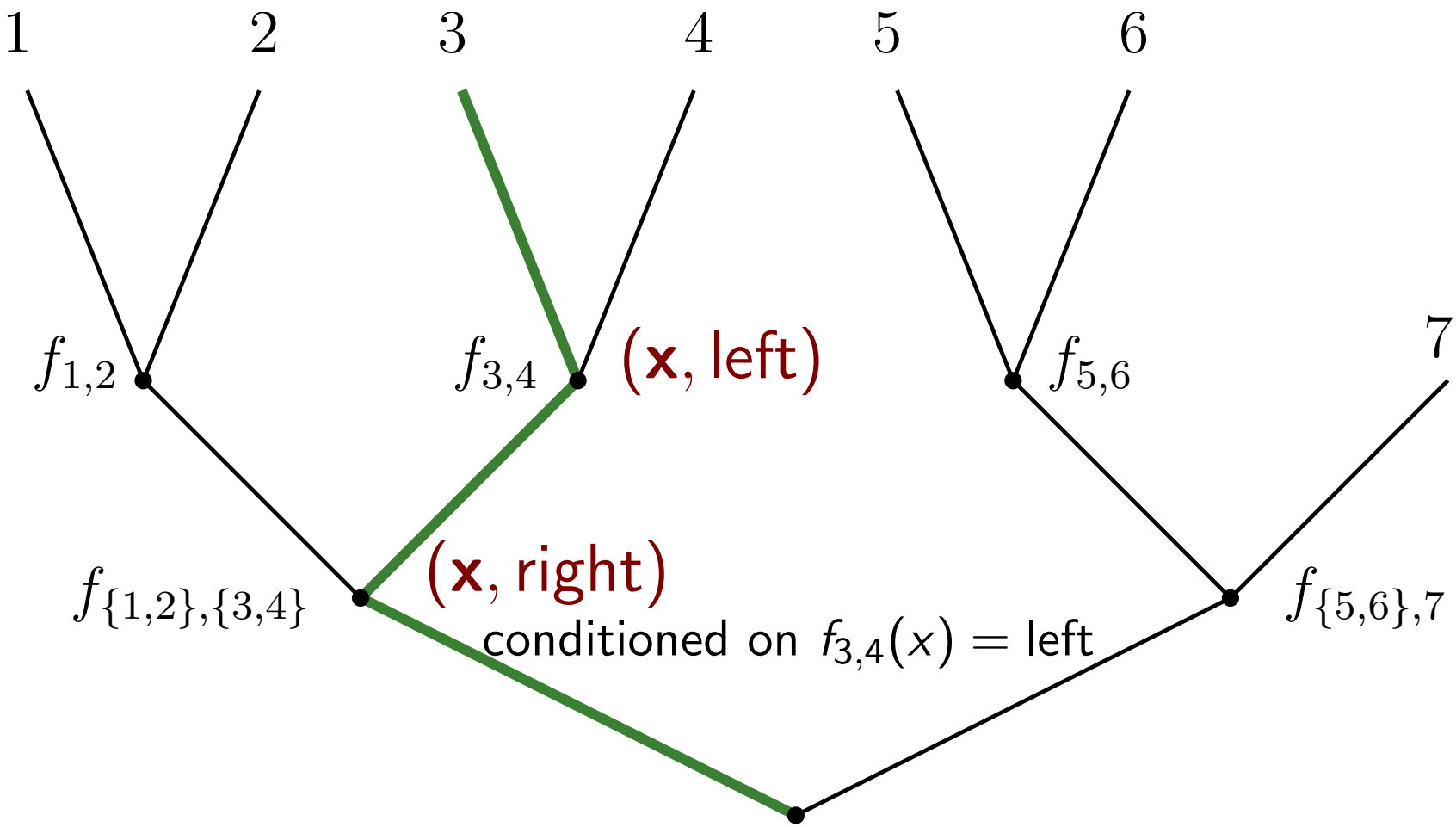

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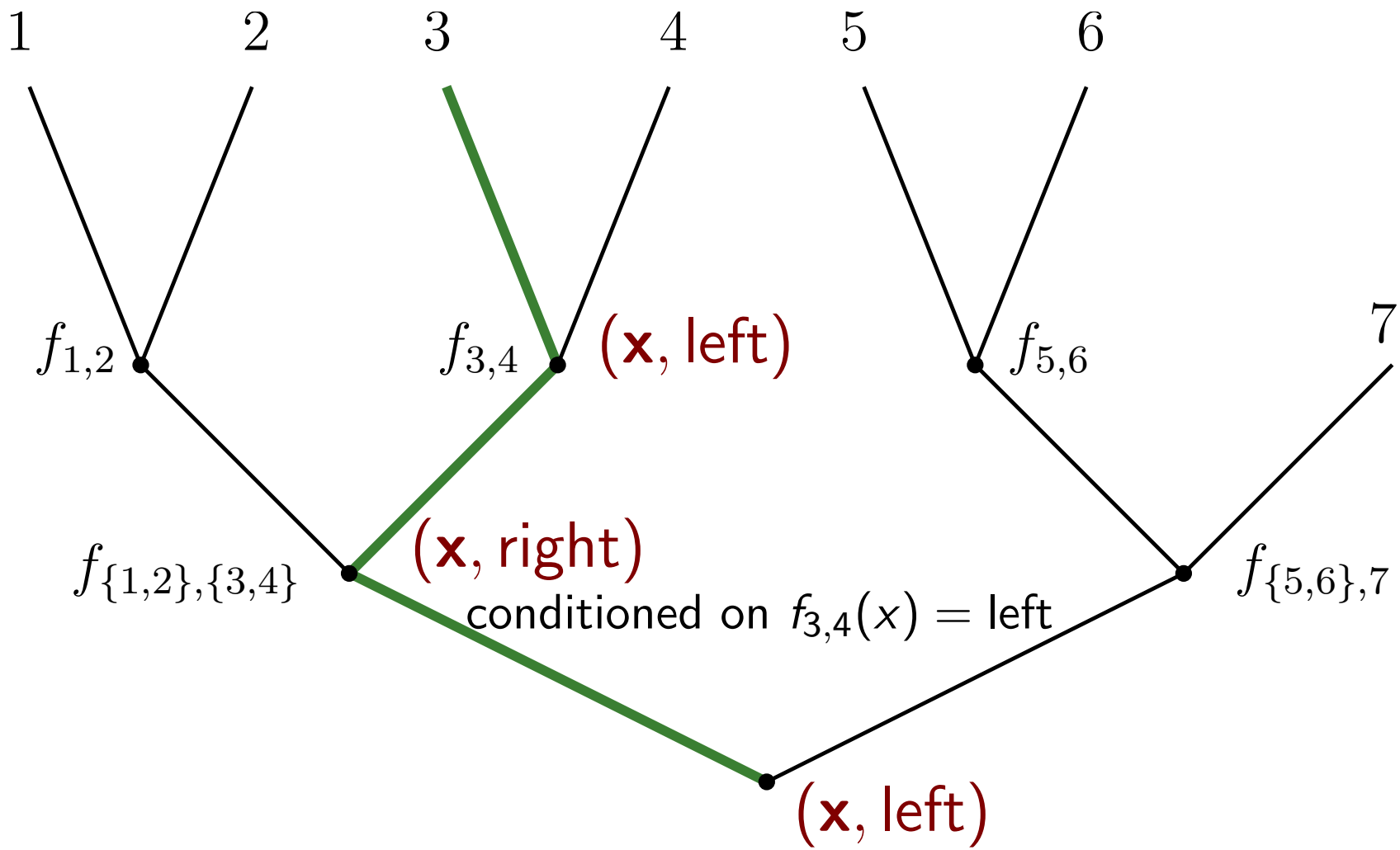


Training

- Important to form the right training sets: Classifiers from the first level are used to filter the distribution of examples reaching the second level, etc.
- Can be composed with either batch or online base learners.

What about with costs?

Cost-sensitive multi-class classification

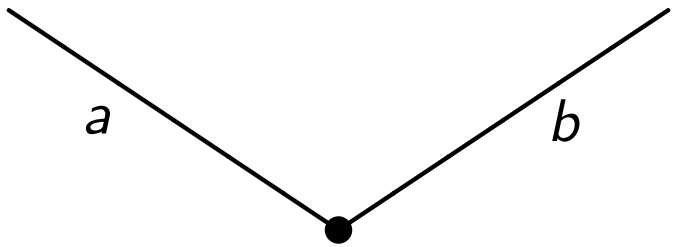
Distribution D over $X \times [0,1]^k$, where a vector in $[0,1]^k$ specifies the cost of each of the k choices.

Find a classifier $h : X \to \{1, \ldots, k\}$ minimizing the expected cost

$$\text{cost}(h, D) = \mathbf{E}_{(x,c)\sim D}[c_{h(x)}].$$

Generalization to the Cost-sensitive Case

To train a non-leaf node on example $(x, c_1, \ldots, c_k)$:



$$\text{Let } y = \begin{cases} \text{left} & \text{if } c_a \leq c_b \\ \text{right} & \text{otherwise} \end{cases}$$

Train on (x, y) with importance weight $|c_a - c_b|$.

Distribution induced at the node

Draw a cost-sensitive example from D, create an importance weighted sample as above.

Analysis

Binary regret:
$\text{reg}_{0/1}(h, D) = \Pr_{(x,y)\sim D}(h(x) \neq y) - \min_{h'} \Pr_{(x,y)\sim D}(h'(x) \neq y)$
Cost-sensitive regret:
$\text{reg}_{CS}(h, D) = E_{(x,y)\sim D}[c_{h(x)}] - \min_{h'} E_{(x,y)\sim D}[c_{h'(x)}]$

Theorem

For all cost-sensitive problems and classifiers at the nodes, the resulting cost-sensitive regret is bounded by average binary regret, times the expected sum of importance weights over the nodes.

$$\text{reg}_{CS}(h_{FT}, D) \leq A(\text{reg}_{0/1}(h, D_{FT}))E_{D_{FT}} \sum_{\text{nodes } n} i_n$$

Alternative bound replaces the sum of importance weights with $k/2$.

The multiclass case

For all multiclass problems and binary classifiers at the nodes, the resulting multiclass regret $\leq$ average binary regret, times $\lceil \log k \rceil$.

Proof: For any example, there is at most one node per level with induced weight 1. Thus tree depth bounds the sum of weights.

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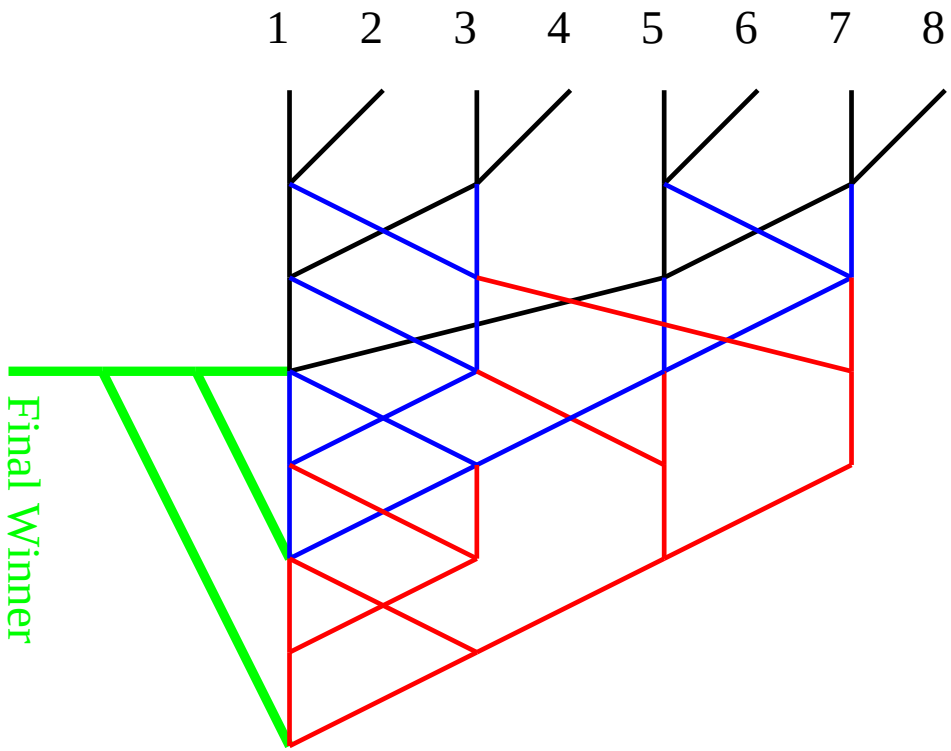
- Using multiple independent single elimination tournaments is of no help since it doesn't affect the average regret of an adversary controlling the binary classifiers.
- ...But we can have $e = O(\log k)$ single elimination tournaments in $O(\log k)$ rounds, with no player playing twice in the same round.

e-elimination tournament

Once an example loses, it moves to the next tournament. Once an example has lost e times, it is eliminated and no longer influences training.

The e winners from the first phase compete in the final single elimination tournament. To win in round i, each player must defeat its opponent 2^{i-1} times.



$e = 3$

Summary of Multiclass results

	Filter Tree	Error Correcting Tournament
MC Computation	$\log k$	$O(\log k)$
MC Regret ratio	$\log k$	5.5
CS Train	k	$O(k \log k)$
CS Test	$\log k$	$O(\log k)$
CS Regret ratio	$\min\{\frac{k}{2}, \sum_n i_n\}$	??

Contextual Bandit Learning in Logarithmic time

Contextual Bandit Classification

Distribution D over $X \times [0,1]^k$, where a vector in $[0,1]^k$ specifies the cost of each of the k choices.

Find a classifier $h : X \to \{1, \ldots, k\}$ minimizing the expected cost

$$\text{cost}(h, D) = \mathbf{E}_{(x,c)\sim D}[c_{h(x)}].$$

given only observations $(x, a, c_a, p_a)^*$.

The Offset Tree for $k = 2$, $p = 0.5$

Suppose $k = 2$ for the moment and let $a \in \{-1, 1\}$. Create binary importance weighted samples according to:

$$\left(x,\ \text{sign}\left(a\left(\frac{1}{2} - c_a \right) \right),\ \left| \frac{1}{2} - c_a \right| \right)$$

x = side information
$\text{sign}\left(a\left(\frac{1}{2} - c_a\right)\right)$ = label
$\left|\frac{1}{2} - c_a\right|$ = importance weight

Denoising Binary Importance Weighting

Theorem

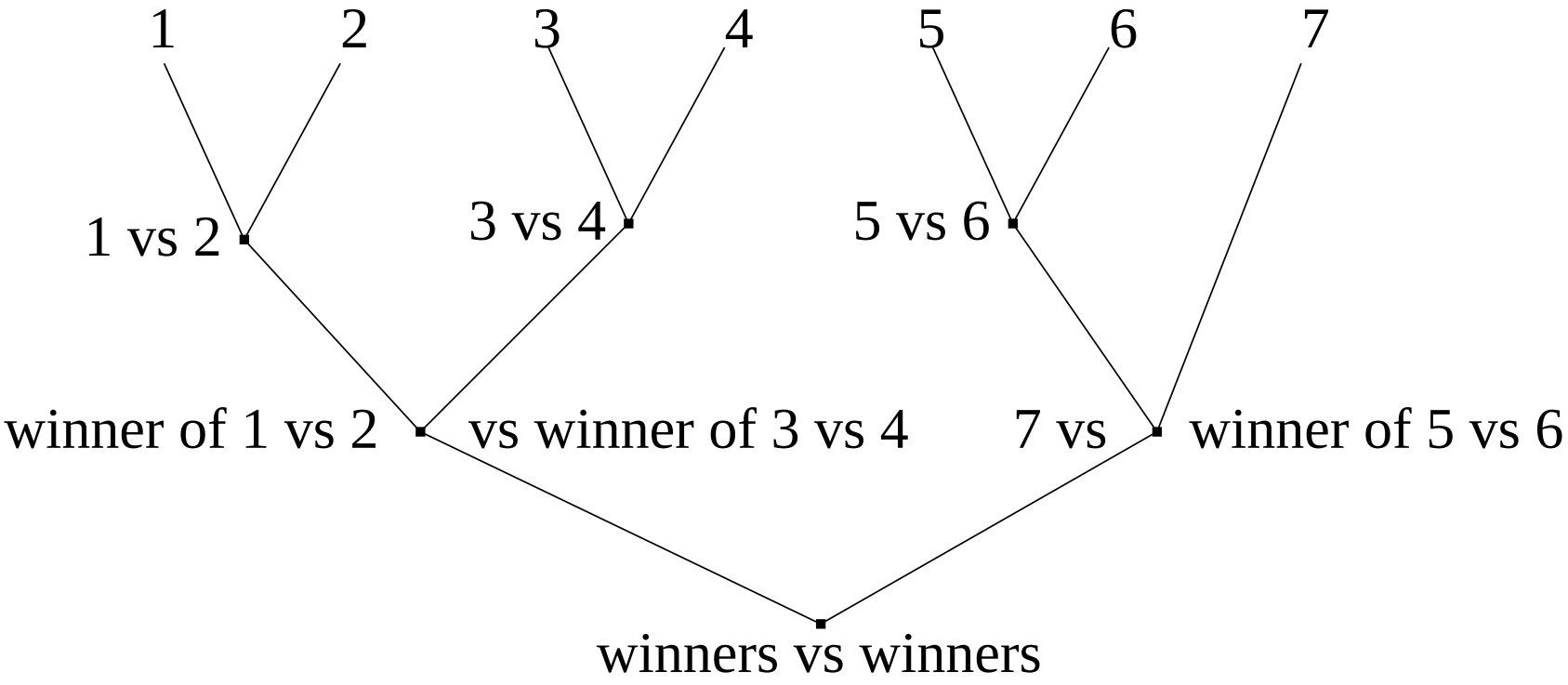
For all Contextual Bandit distributions D with $k = 2$, for all binary classifiers b:

$$\text{policy regret} \leq \text{reg}_{0/1}(b, D_{OT}).$$

The induced problem is often noisy. This trick reduces the maximum noise, giving a factor of 2 improvement in the upper bound.

$\frac{1}{2}$ = minimax value of the median reward. Plugging in the actual median is always better.

Denoising for $k > 2$ arms



Use the same construction at each node. Internal nodes only get an example if all leaf-wards nodes agree with the label (Filtering trick).

Denoising with k arms

D_{OT} = Induced Binary classification problem
b = the classifier which predicts based on both x and the choice of binary problem according to D_{OT}.

Theorem

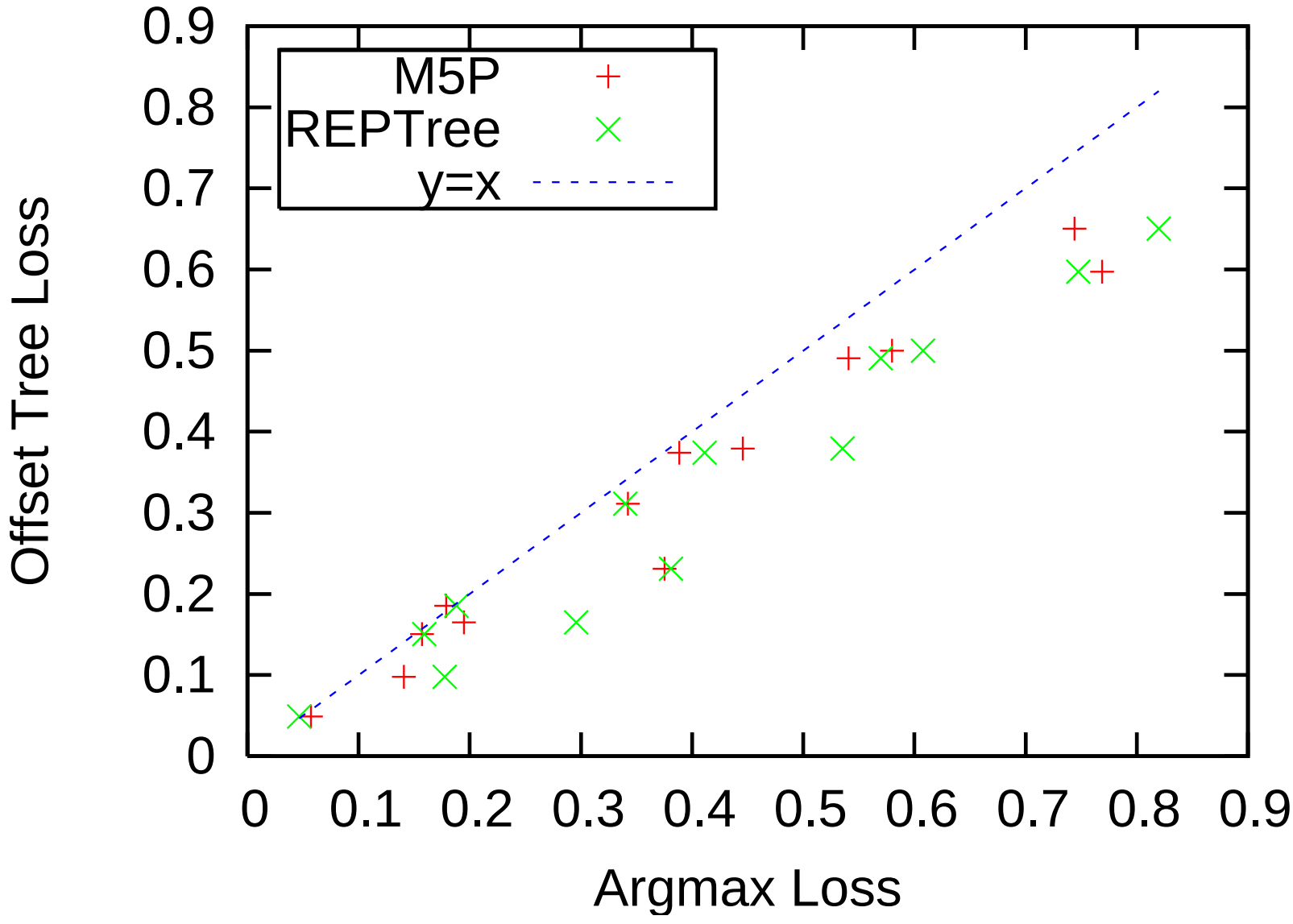
For all k-choice D, binary classifiers b:

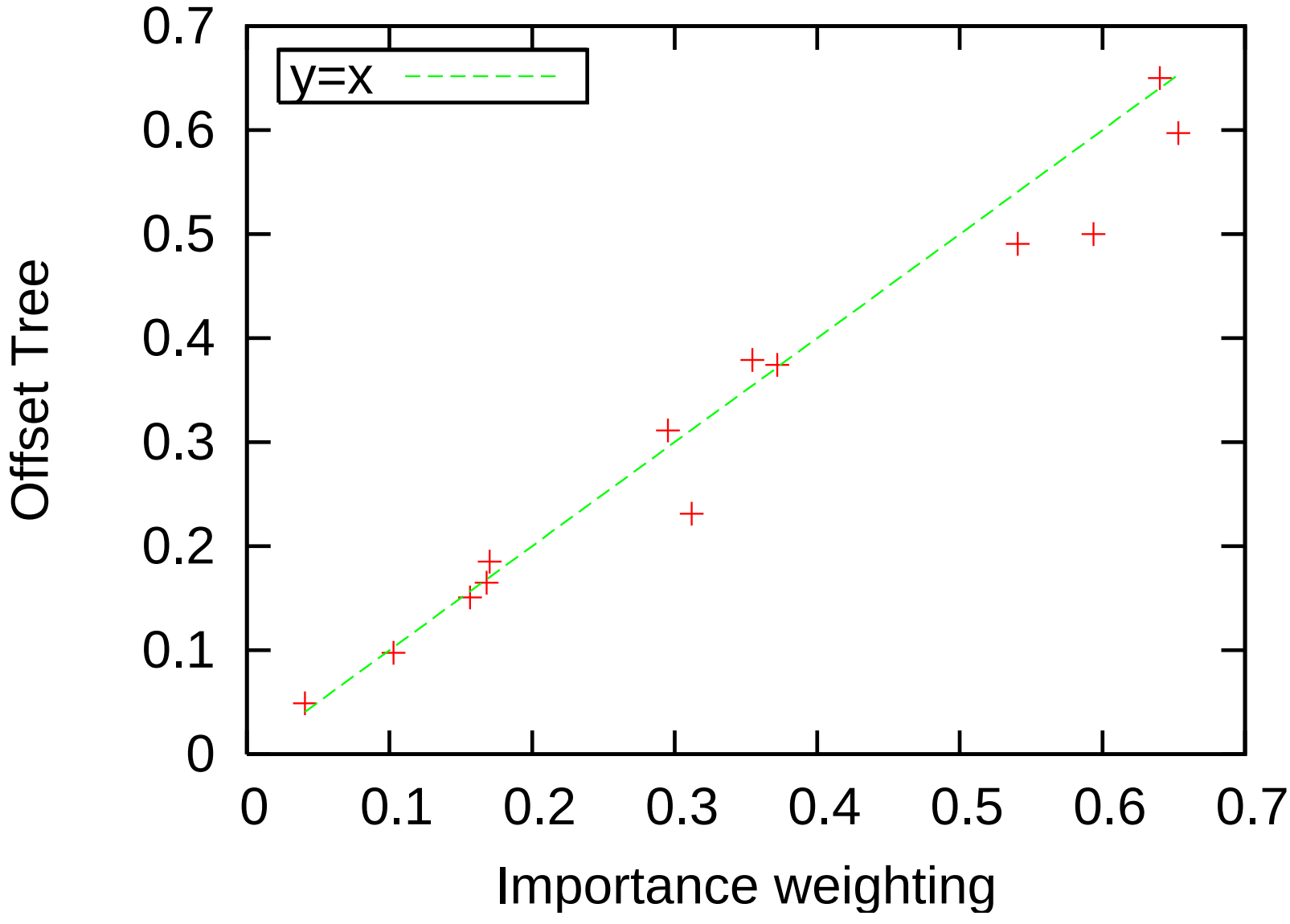
$$\text{policy regret} \leq (k-1)\,\text{reg}_{0/1}(b, D_{OT}).$$

A Comparison of Approaches

Algorithm	Policy Regret Bound
Argmax	$\sqrt{2k \operatorname{reg}_{0/1}(s, D_{AR})}$
Importance Weighted	$4k \operatorname{reg}_{0/1}(b, D_{IW})$
Offset Tree	$(k-1) \operatorname{reg}_{0/1}(b, D_{OT})$

How do you expect things to work, experimentally?


M5P
REPTree
y=x
Offset Tree Loss
Argmax Loss
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9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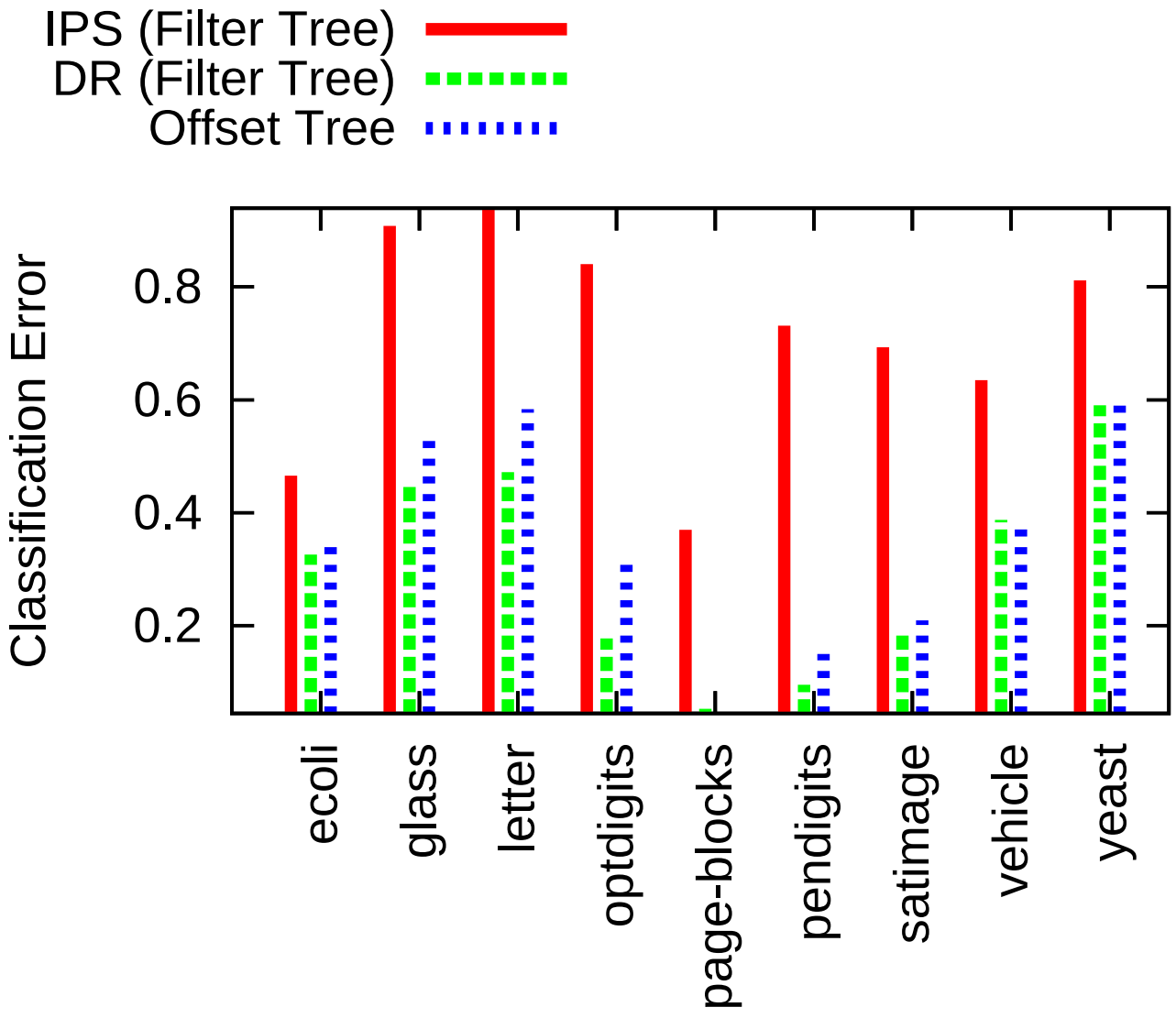
Compare with Double Robust approaches empirically



Double Robust is exponentially slower, but often a bit better.

The last problem

A dataset:

1. $40 * 10^6$ (ad,webpage) pairs ($+ 10 * 10^6$ pairs in test set)
2. $\sim 10^7$ unique webpages
3. $\sim 10^6$ unique ads
4. $\sim 10^6$ unique features.

A problem:
Predict: Pr(ad|webpage) for a given (ad, webpage) pair.

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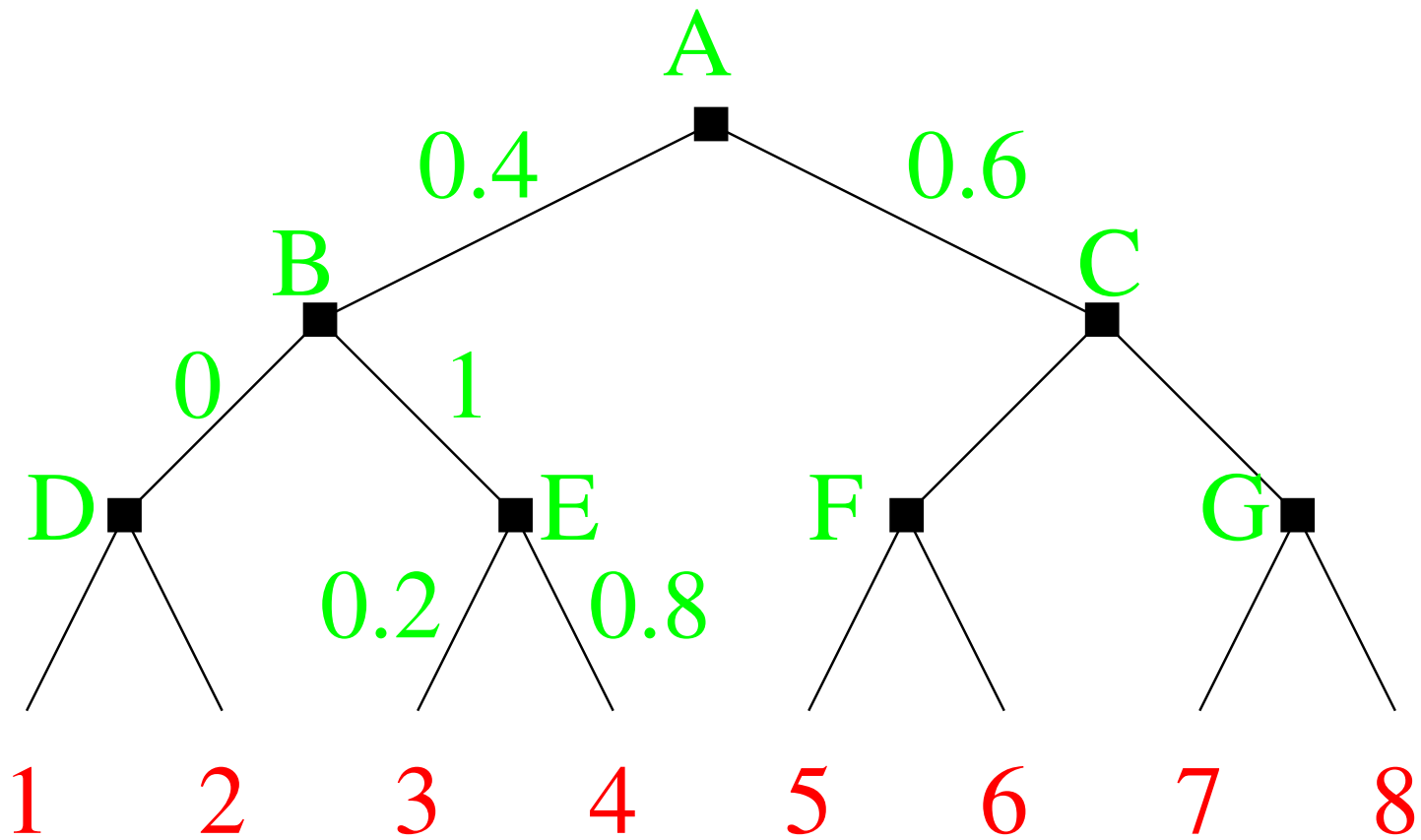
Conditional Probability Estimation

Distribution D over $X \times Y$, where $Y = \{1, \ldots, k\}$.

Find a Probability estimator $h : Y \times X \to [0, 1]$ minimizing squared loss

$$\ell(h, D) = E_{(x,y)\sim D}[(h(y|x) - y)^2]$$

One approach: A Conditional Probability Tree



Every node predicts $P(\text{ad is to the right}|\text{page})$.
Number of nodes to evaluate: $\log_2 |\text{ads}|$.

$$\hat{P}(\text{ad 4}|\text{page}) = (1 - \text{A}(\text{page}))\text{B}(\text{page})\text{E}(\text{page})$$

$$= 0.4 * 1 * 0.8 = 0.32$$

But is the prediction too unstable to work?

Basic fact: minimizer of squared loss = correct probability.
Theorem: For any distribution P, Any regressors Q, and any pair (x, y),

$$(P_Q(y|x) - P(y|x))^2$$

$$\leq \text{depth}^2 E_{i \in \text{Path}}(Q_i(\text{right}_i(y)|\text{page}) - P_i(\text{right}_i(y)|\text{page}))^2$$

$$= \text{depth}^2(\text{average binary regret})$$

A similar statement for one-against-all approaches replaces depth^2 with $|\text{ads}|^2$. This is much worse, although it often is not realized in practice.

Proof

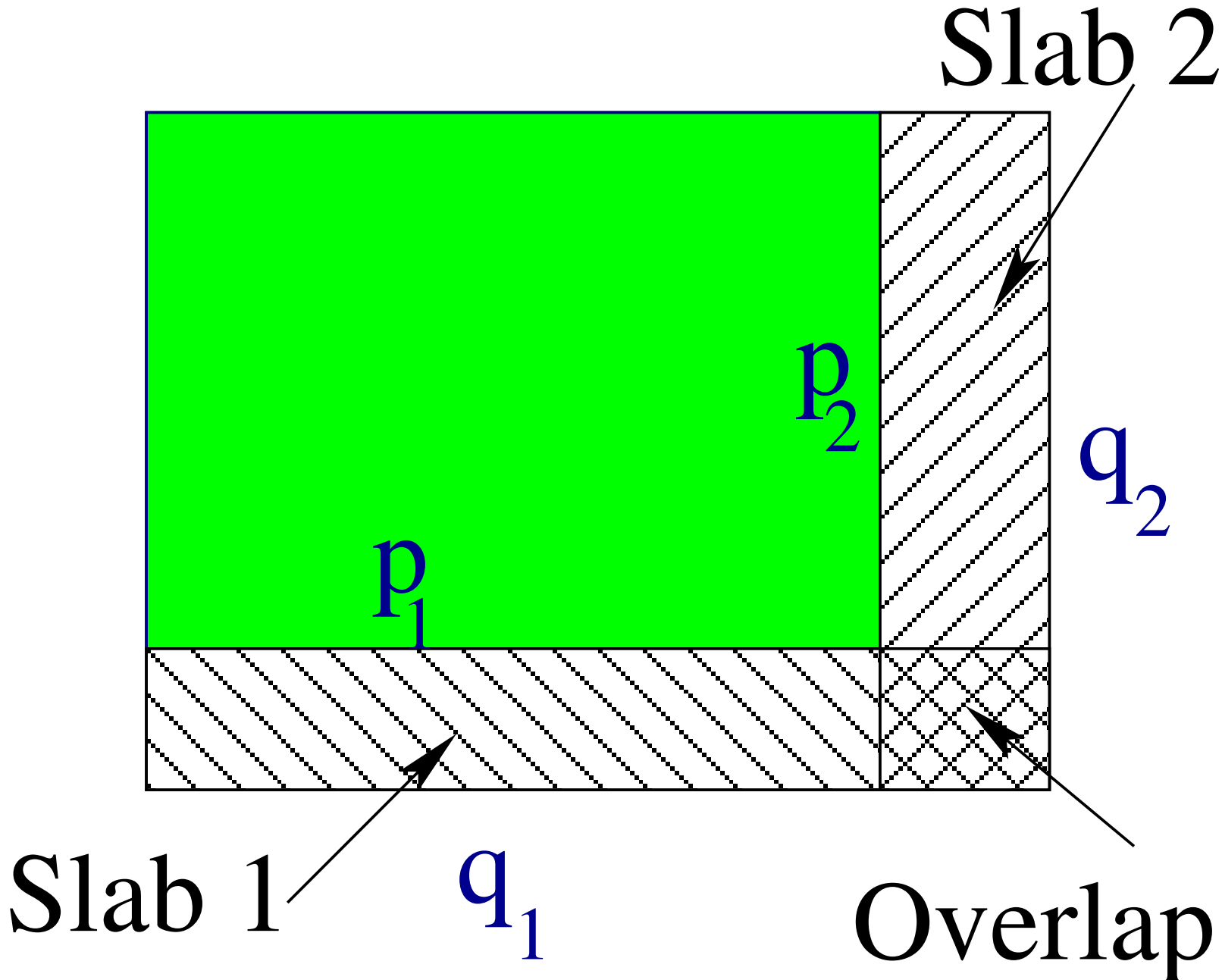
Let $q_i = Q_i(\text{right}_i(y)|x)$ and $p_i = P_i(\text{right}_i(y)|x)$.
$|P_Q(y|x) - P(y|x)| \leq \prod_i \max\{q_i, p_i\} - \prod_i \min\{q_i, p_i\}$
Using geometry, $\prod_i \max\{q_i, p_i\}$, $\prod_i \min\{q_i, p_i\}$ = hypercube volumes.
Small hypercube + slabs $\geq$ big hypercube.

$$\prod_i \max\{q_i, p_i\} - \prod_i \min\{q_i, p_i\} \leq \sum_i |q_i - p_i| \prod_{j \neq i} \max\{q_j, p_j\}$$

$$\leq \sum_i |q_i - p_i|$$

... then apply Jensen's inequality.

The hypercube arbument in 2-d



How do we assign labels to leaves?

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Random left/right decisions: 0.7742 ± 0.0006 (Uniform over ~ 8.32 ads)

Surprisingly good, but surely we can do better than random?

A better than random approach

When a new label is encountered, starting at the root and descend, send the label right if:

$$(1-\alpha)\left(2Q_i(\text{page})-1\right)+\alpha\log_2\frac{|\text{ads left}|+1}{|\text{ads right}|+1}>0$$

Theorem: For all $\alpha\in[0.5,1]$, for all binary regressors,

$$\text{depth}\leq 3+\frac{\log_2|\text{ads}|}{\log_2\left(1+2^{\frac{\alpha-1}{\alpha}}\right)}$$

Proof: Requires a bit of work because of discretization effects.

RAM problems

How do you store and use 10^6 binary regressors each with $\sim 10^6$ features?

We don't have $4 * 10^{12}$ bytes of RAM!

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Hashing

Conditional Probability Tree on Ad dataset

1. Training time $\sim$ 1 hour on a single machine.
2. RAM usage $\sim$ 100MB.
3. Squared loss 0.7632 ± 0.0006 (equivalent to uniform random over 7.91 ads).

Conditional Probability Tree on RCV1

(Full) RCV1 has:

1. 103 classes
2. 780K examples
3. $\sim 2^{16}$ features

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	Conditional Probability Tree	One Against All
Computation	$108s$	$2300s$
Squared Loss	0.56 ± 0.012	0.55 ± 0.012

Losing a small amount of prediction performance for a significant gain.

Choose your baseline

1. Relative to $O(k)$ ML techniques, giant improvements in computation + sometimes better/sometimes worse performance.
2. Relative to index+rank approaches, perhaps a significant improvement in performance.

This area is immature: there are plausibly several theses to write.

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